Comments On The Fibonacci Sequences In Finite Groups

Ömür DEVECİ & Erdal KARADUMAN

Atatürk University , Department of Mathematics Faculty of Science

25240 Erzurum / TURKEY

E-mail: odeveci36@hotmail.com, eduman@atauni.edu.tr

Abstract

In the work of Knox, Steven W. (1992), the claim is made that “A $k$-nacci sequence in a finite group is simply periodic [11].” We provide an example to demonstrate that the claim is false.

1. Introduction

The study of Fibonacci sequences in groups began with the earlier work of Wall [12] where the ordinary Fibonacci sequences in cyclic groups were investigated. In the mid eighties, Wilcox extended the problem to abelian groups [13]. Prolific co-operation of Campbell, Doostie and Robertson expanded the theory to some finite simple groups [5]. Aydın and Smith proved in [2] that the lengths of ordinary 2-step Fibonacci sequences are equal to the lengths of ordinary 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and a prime exponent. The theory has been generalized in [7,8] to the ordinary 3-step Fibonacci sequences in finite nilpotent groups. Then, it is shown in [1] that the period of 2-step general Fibonacci sequence is equal to the length of fundamental period of the 2-step general Fibonacci sequence constructed by two generating elements of the group of exponent $p$ and nilpotency class 2. Karaduman and Yavuz showed that the periods of the 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 5 and a prime exponent are $p.k(p)$, for $2 < p \leq 2927$, where $p$ is prime and $k(p)$ is the periods of ordinary 2-step Fibonacci sequences[10]. Knox proved that periods of $k$-nacci ($k$-step Fibonacci) sequences in
dihedral group were equal to \(2k + 2\) [11]. Recently, the works have been done on Fibonacci sequences. See, for example, [3,4,6,9].

A \(k\)-nacci sequence in a finite group is a sequence of group elements \(x_0, x_1, x_2, \ldots, x_n, \ldots\) for which, given an initial (seed) set \(x_0, x_1, x_2, \ldots, x_{j-1}\), each element is defined by

\[
x_n = \begin{cases} 
x_0 x_1 \cdots x_{n-1} & \text{for } j \leq n < k \\
x_{n-k} x_{n-k+1} \cdots x_{n-1} & \text{for } n \geq k 
\end{cases}
\]

We also require that the initial elements of the sequence, \(x_0, x_1, x_2, \ldots, x_{j-1}\), generate the group, thus forcing the \(k\)-nacci sequence to reflect the structure of the group. The \(k\)-nacci sequence of a group generated by \(x_0, x_1, x_2, \ldots, x_{j-1}\) is denoted by \(F_k(G; x_0, x_1, \ldots, x_{j-1})\) and its period is denoted by \(P_k(G; x_0, x_1, \ldots, x_{j-1})\).

**Definition 1.** A 2-step Fibonacci sequence of a group elements is called a Fibonacci sequence of a finite group.

**Definition 2.** A finite group \(G\) is \(k\)-nacci sequenceable if there exists a \(k\)-nacci sequence of \(G\) such that every element of the group appears in the sequence.

**Definition 3.** A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called period of the sequence. For example, the sequence \(a, b, c, d, e, b, c, d, e, b, c, d, e, \ldots\) is periodic after the initial element \(a\) and has period 4.

**Definition 4.** A sequence of group elements is simply periodic with period \(k\) if the first \(k\) elements in the sequence form a repeating subsequence. For example, the sequence \(a, b, c, d, e, f, g, a, b, c, d, e, f, g, a, b, c, d, e, f, g, \ldots\) is simply periodic with period 7.

The following appears in [11, Theorem 1].

**Theorem 5**: A \(k\)-nacci sequence in a finite group is simply periodic [11].
**Definition 5.** The *binary polyhedral group* \(<l,m,n>\), for \(l,m,n > 1\) is defined by the presentation
\[
< x, y, z : x^l = y^m = z^n = xyz >.
\]

Now, we will give an example satisfying the condition of [11, Theorem 1], but the 2-nacci sequence in a finite group is not to be necessary *simply* periodic.

**Example 7.** Let us consider the *binary polyhedral group* \(<2,2,n>\), for \(n > 2\), defined by the presentation
\[
< x, y, z : x^2 = y^2 = z^n = xyz >.
\]

The order the group defined by this presentation is \(4n\) and the order of \(z\) is \(2n\) and the orders of \(x\) and \(y\) are 4. Thus, from relations in the group we have
\[
\begin{align*}
x &= yz, \\
y &= zx, \\
z &= y^3x = yx^3.
\end{align*}
\]

Using definition of a \(k\)-nacci sequence in a finite group, we obtain 2-nacci sequence in the group defined by this presentation as follow:
\[
\begin{align*}
x_0 &= x, \\
x_1 &= y, \\
x_2 &= z, \\
x_3 &= yz = x, \\
x_4 &= zx = y, \\
x_5 &= xy, \\
x_6 &= yxy, \\
x_7 &= xyxyxy = x^4 y = y = x_1, \\
x_8 &= yxyxyy = yx^3 = z = x_2, \\
x_9 &= yz = x = x_3, \\
x_{10} &= zx = y = x_4, \\
x_{11} &= xy = x_5, \\
x_{12} &= yxy = x_6, \\
&\vdots
\end{align*}
\]
It is clear that $2$-nacci sequence in the group $<2, 2, n>$ is periodic with period 6 but not simply periodic.

REFERENCES


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